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1984 J. Phys. A: Math. Gen. 17 L251

(http://iopscience.iop.org/0305-4470/17/5/006)

J. Phys. A: Math. Gen. 17 (1984) L251–L255. Printed in Great Britain

LETTER TO THE EDITOR

General solutions of large-n renormalisation group equations

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Received 3 January 1984

Abstract. We solve the recently obtained large-n differential renormalisation-group equations for the Ginsburg–Landau–Wilson and time-dependent Ginsburg–Landau models. We determine the fixed points of our solutions and make brief comparisons with solutions determined previously from generating functions of nonlinear scaling fields.

The large-n limit of the renormalisation-group (RG) recursion relations has been extensively studied in both the static (Ma 1973, 1974, 1976) and dynamical (Szépfalusy and Tél 1980a, b) settings. Although much effort has been directed towards determining and characterising fixed points of the RG transformation, the existence of generating functions for nonlinear scaling fields (Ma 1974) has facilitated the construction of approximate global solutions of the large-n recursion relations (Zannetti and Di Castro 1977, Szépfalusy and Tél 1980b). On the other hand, differential formulations of the RG (Wegner and Houghton 1973, Wilson and Kogut 1974, Vvedensky *et al* 1983) generally provide the most suitable framework for the study of global RG trajectories (Nicoll *et al* 1975), particularly in the large-n limit where the differential RG (DRG) transformations become quasi-linear partial differential equations (Wegner and Houghton 1973, Nicoll *et al* 1976 Busiello *et al* 1981, 1983, Vvedensky 1984). In this letter we present general solutions of large-n DRG equations for the Ginsburg–Landau–Wilson (GLW) model and the time-dependent Ginsburg–Landau (TDGL) model with relaxational dynamics. We briefly compare these exact solutions with those obtained from generating functions of nonlinear scaling fields (Zannetti and Di Castro 1977, Szépfalusy and Tél 1980b) but we save more detailed comparison with earlier work for a future paper.

We begin by considering an isotropic d-dimensional system ($d>2$) characterised by an n-component order parameter $\psi_i(x)$, $i=1,\ldots,n$ and with the usual reduced GLW Hamiltonian:

$$\mathscr{H}=\int \mathrm{d}\boldsymbol{x}[(\nabla\boldsymbol{\psi})^2+H(\psi^2)] \tag{1}$$

where

$$H(\psi^2)=\sum_{p=1}^{\infty} u_{2p}(\psi^2)^P \tag{2}$$

and where $\psi^2=\boldsymbol{\psi}\cdot\boldsymbol{\psi}$. Defining $H'(\psi^2)=\mathrm{d}H/\mathrm{d}(\psi^2)$ and $x=\psi^2/N_c$, where $N_c=nS_d/2(d-2)(2\pi)^d$ and S_d is the surface area of a unit d-sphere, the large-n DRG equation is given in terms of $t(x) \equiv H'(xN_c)$ (Busiello *et al* 1981, Vvedensky 1984):

$$\partial t/\partial l = 2t + (2-d)[x - 1/(1+t)](\partial t/\partial x) \tag{3}$$

subject to the initial condition

$$t(0, x) = \sum_{p=1}^{\infty} p u_{2p}(0)(xN_c)^{p-1} \tag{4}$$

and to the critical fixed-point equation (Busiello *et al* 1981)

$$\begin{aligned} x &= 1 + \tfrac{1}{2}(d-2)t^* \int_0^1 \frac{z^{(2-d)/2}\,\mathrm{d}z}{1+zt^*} \\ &= 1 + \frac{d-2}{4-d}\, t^*{}_2F_1(1, 2-d/2; 3-d/2; -t^*) \end{aligned} \tag{5}$$

where $t^*(x) = \lim(l \to \infty) t(l, x)$ and ${}_2F_1(\alpha, \beta; \gamma; z)$ is the usual hypergeometric function (Gradshteyn and Ryzhik 1965).

The general solution to (3)–(5) is easily determined by the method of characteristics (Courant and Hilbert 1962) and may be written in the implicit form

$$t = \mathrm{e}^{2l}\mathscr{G}\left[2t^{(2-d)/2}\left(x - 1 - \frac{1}{2}(d-2)t \int_0^1 \frac{z^{(2-d)/2}\,\mathrm{d}z}{1+zt}\right)\right] \tag{6}$$

where in view of (4) $\mathscr{G}$ is determined by the power series expansion

$$\mathscr{G}(z) = \sum_{r=0}^{\infty} a_r z^r. \tag{7}$$

The general solution (6), (7) has the following properties.

(i) The expansion coefficients a_n in (7) are determined by the initial conditions (4). Thus, for *any* choice of the a_n, (6) is an exact solution of (3). For trajectories on the critical surface ($t(l, 1) = 0$) we have that $a_0 = 0$.

(ii) There are two critical fixed points of (7), the Gaussian fixed point $t^* = 0$ and the spherical fixed point (5), as well as a non-critical fixed point $t^* \to \infty$, the infinite Gaussian fixed point (Nicoll *et al* 1975).

(iii) There are several formal similarities between the general solution (6), (7) and the exact solution obtained by Zannetti and Di Castro (1977), from generating functions of nonlinear scaling fields. However, while (6) is an exact solution of (3) even if only a finite subset of the $\{a_n\}$ are non-vanishing, the corresponding exact solution in the representation of Zannetti and Di Castro (1977) necessarily involves an infinite number of terms. Evidently the representation (6) is the more appropriate for building general solutions to (3) by including successively more terms in (7).

For the TDGL model we begin with the generalised Langevin equation

$$\dot{\psi}_i(xt) = -\Gamma(x)[\delta\mathscr{H}/\delta\psi_i(xt)] + \eta_i(xt) \tag{8}$$

where $\mathscr{H}$ is given by (1) and (2), Γ is a transport coefficient, and the n-component stochastic term $\boldsymbol{\eta}$ has a Gaussian white-noise spectrum. Introducing the field $\boldsymbol{\phi}$ conjugate to $\boldsymbol{\eta}$, the generalised action for the process (8) is given by (Szépfalusy and Tél 1980a)

$$\mathscr{A} = \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}t[\mathrm{i}\boldsymbol{\phi} \cdot (\nabla^2\boldsymbol{\psi} + \Gamma^{-1}\dot{\boldsymbol{\psi}}) + \Gamma^{-1}\mathrm{i}\boldsymbol{\phi} \cdot \mathrm{i}\boldsymbol{\phi} + A(\boldsymbol{\psi} \cdot \boldsymbol{\psi}, \mathrm{i}\boldsymbol{\phi} \cdot \boldsymbol{\psi})] \tag{9}$$

where

$$A(\boldsymbol{\psi}\cdot\boldsymbol{\psi}, \mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi}) = \sum_{p=1}^{\infty}\sum_{q=1}^{p} u_{2p,2q}(\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi})^{q}(\boldsymbol{\psi}\cdot\boldsymbol{\psi})^{p-q}. \tag{10}$$

Defining $A_1'(\boldsymbol{\psi}\cdot\boldsymbol{\psi}, \mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi}) = \partial A/\partial(\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi})$ and $A_2'(\boldsymbol{\psi}\cdot\boldsymbol{\psi}, \mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi}) = \partial A/\partial(\boldsymbol{\psi}\cdot\boldsymbol{\psi})$, and introducing the variables $x = \mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi}(d+z-2)/(d-2)N_c$ and $y = \boldsymbol{\psi}\cdot\boldsymbol{\psi}/N_c$, the large-$n$ dynamical DRG equations are expressed in terms of $t_i(x, y) = A_i'[x(d-2)N_c/(d+z-2), N_c y]$, $i = 1, 2$ (Busiello *et al* 1983, Vvedensky 1984):

$$\partial t_i/\partial l = \lambda_i t_i + (2-d-z)[x + F(t_1, t_2)](\partial t_i/\partial x) + (2-d)[y - G(t_1, t_2)](\partial t_i/\partial y) \tag{11}$$

where

$$F(t_1, t_2) = (1+t_1)G(t_1, t_2), \qquad G(t_1, t_2) = [(1+t_1)^2 - 2t_2]^{-1/2} \tag{12}$$

with $\lambda_1 = 2$, $\lambda_2 = 2+z$ and $z = 4$ (resp. 2) if the order parameter is conserved (resp. not conserved). The initial conditions are determined in analogy with (4):

$$\begin{aligned} t_1(0, x, y) &= \sum_{p=1}^{\infty}\sum_{q=1}^{p} p u_{2p,2q}(0) N_c^{p-1}\left(\frac{d-2}{d+z-2}\right)^{q-1} x^{q-1} y^{p-q} \\ t_2(0, x, y) &= \sum_{p=1}^{\infty}\sum_{q=1}^{p} (p-q) u_{2p,2q}(0) N_c^{p-1}\left(\frac{d-2}{d+z-2}\right)^{q} x^{q} y^{p-q-1} \end{aligned} \tag{13}$$

and the fixed-point condition has been determined by Szépfalusy and Tél (1980a). In the case when the initial action is determined from (8), we have that only the $u_{2p,2}(0)$ and $u_{2p,2p}(0)$ are non-vanishing.

Although the general solution to (11)–(13) may in principle be obtained by the method of characteristics (Courant and Hilbert 1962), a much simpler and more direct procedure is to construct a general solution in analogy to (6):

$$t_i = \mathrm{e}^{\lambda_i l}\mathscr{G}_i(2xt_1^{(2-d-z)/2} + f(t_1, t_2), (2+z)(y-1)t_2^{(2-d)/(2+z)} + g(t_1, t_2)) \tag{14}$$

where the $\mathscr{G}_i$ are specified by

$$\mathscr{G}_i(z_1, z_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} a_{i;rs} z_1^{r-s} z_2^{s} \tag{15}$$

and we choose f and g to be solutions of

$$\begin{aligned} \lambda_1 t_1\,\partial f/\partial t_1 + \lambda_2 t_2 \partial f/\partial t_2 &= \lambda_1(2-d-z)t_1^{(2-d-z)/\lambda_1} F \\ \lambda_1 t_1\,\partial g/\partial t_1 + \lambda_2 t_2 \partial g/\partial t_2 &= \lambda_2(2-d)t_2^{(2-d)/\lambda_2}(1-G) \end{aligned} \tag{16}$$

with the boundary conditions

$$f(t_1, 0) = 0, \qquad g(0, 0) = 0. \tag{17}$$

The solutions to (16) and (17) may be written in the form

$$\begin{aligned} f(t_1, t_2) &= \lambda_1(2-d-z)t_1^{(2-d-z)/\lambda_1}\int_0^1 \mathrm{d}s\, s^{-1} s^{2-d-z}[F(s^{\lambda_1}t_1, s^{\lambda_2}t_2) - 1] \\ g(t_1, t_2) &= \lambda_2(2-d)t_2^{(2-d)/\lambda_2}\int_0^1 \mathrm{d}s\, s^{-1} s^{2-d}[1 - G(s^{\lambda_1}t_1, s^{\lambda_2}t_2)]. \end{aligned} \tag{18}$$

Our general solution (14)–(18) has the following properties.

(i) The expansion coefficients $a_{i,rs}$ are determined by the initial conditions (13). The condition $t_2(0, y) = 0$ (Szépfalusy and Tél 1980a) requires that $a_{2,rr} = 0$ for all r. Moreover, for trajectories on the critical surface, where $t_1(0, 1) = 0$, we must have $a_{1,00} = 0$.

(ii) The non-trivial fixed point of (14)–(18) is determined by the coupled equations

$$
\begin{aligned}
x &= (d+z-2)\int_0^1 \mathrm{d}s\, s^{-1} s^{2-d-z}[F(s^2 t_1, s^{2+z} t_2) - 1] \\
y &= 1 + (d-2)\int_0^1 \mathrm{d}s\, s^{-1} s^{2-d}[1 - G(s^2 t_1, s^{2+z} t_2)]
\end{aligned}
\qquad (19)
$$

which are equivalent to the fixed-point equations obtained by Szépfalusy and Tél (1980a) under the transformation $s \to s^{-1}$.

(iii) As in the case of our solution (6), the general solution (14)–(18) is a much more convenient representation than that obtained from generating functions of nonlinear scaling fields (Szépfalusy and Tél 1980b).

The possibility of constructing general solutions involving only one arbitrary function for each t_i results from the fact that the large-n RG equations are first-order quasi-linear partial differential equations. This in turn is a direct result of the ordering of the variables, namely, $\psi = O(n)$ and $u_{2p} = O(n^{1-p})$ in the static case (Ma 1973) and similarly in the dynamical case (Szépfalusy and Tél 1980a). On the other hand, the ordering of Nicoll *et al* (1976), based upon a slightly different interpretation of the large-n limit, yields a second-order quasi-linear equation with the same critical indices as (3). The relationship between these two large-n forms of the RG will be discussed in a future paper.

Finally, we should like to point out that exact differential formulations of real-space RG transformations for the two-dimensional triangular Ising model (Hilhorst *et al* 1978, 1979) and the d-dimensional Gaussian model (Yamazaki *et al* 1980) also yield systems of coupled quasi-linear first-order equations. An interesting feature of these equations is that they resemble classical equations of motion, which is a point of view that has been pursued by Busiello *et al* (1983) for the system (11). General solutions of these real-space DRG equations would be a useful complement to the general solutions presented here, though their determination appears not to be straightforward.

This work is based upon research carried out while the author was a NATO Postdoctoral Fellow (administered by the US National Science Foundation) at University College London. Support at Imperial College by the British Petroleum Venture Research Unit is also gratefully acknowledged.

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